METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 2	
Code : *MAT 120* Acad.Year : *2014-2015* Semester : *FALL* Date : *05.12.2014* Time : *15:40* Duration : *110 min*	Last Name: KEY Name : Student # : Signature : 7 QUESTIONS ON 7 PAGES TOTAL 100 POINTS

1. (10)	2. (10)	3. (15)	4. (15)	5. (30)	6. (10)	7. (10)	

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(10pts)* Find an equation of the tangent plane to the surface $2x^2 + 3y^2 + \sin z = 1$ at the point $(\frac{1}{\sqrt{2}}, 0, \pi)$.

The point $P(\frac{1}{\sqrt{2}}, 0, \pi)$ is on the surface, for $2\left(\frac{1}{\sqrt{2}}\right)^2 + 3\cdot 0^2 + \sin(\pi) = 1$. If $f(x,y,z) = 2x^2+3y^2+\sin(z)$, we have

$\nabla f = \langle 4x, 6y, \cos(z)\rangle$ and $(\nabla f)_P = \langle \frac{4}{\sqrt{2}}, 0, -1\rangle$.

Taking into account that $(\nabla f)_P$ is a normal vector to the tangent plane at P, we derive that

$$\frac{4}{\sqrt{2}}\left(x - \frac{1}{\sqrt{2}}\right) - (z-\pi) = 0$$

is the plane equation sought.

$$\boxed{z = \frac{4}{\sqrt{2}}x + \pi - 2}$$

2. *(5+5=10pts)* Consider the function $f(x,y) = \sqrt{xy} + 24$

(a) Find the directional derivative of f at the point $P(1,1)$ in the direction of the vector $v = \langle 3, 4 \rangle$.

The direction of the vector $\vec{v}$ is $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

Further, $\nabla f = \langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \rangle$ and $(\nabla f)_P = \langle \frac{1}{2}, \frac{1}{2} \rangle$.

Since $(D_{\vec{u}} f)_P = (\nabla f)_P \cdot \vec{u}$, we obtain that

$(D_{\vec{u}} f)_P = \langle \frac{1}{2}, \frac{1}{2} \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$,

that is,

$$\boxed{(D_{\vec{u}} f)_P = \frac{7}{10}}$$

(b) At the point $Q(1,4)$ in what direction does f have the highest rate of change? What is the rate of change in this direction.

The function f increases most rapidly in the direction of $(\nabla f)_Q$ and its rate of change equals to $\|(\nabla f)_Q\|$. Thus

$(\nabla f)_Q = \langle \frac{4}{2\sqrt{4}}, \frac{1}{2\sqrt{4}} \rangle = \langle 1, \frac{1}{4} \rangle$, $\boxed{\|(\nabla f)_Q\| = \frac{\sqrt{17}}{4}}$

and $\boxed{\vec{u} = \frac{(\nabla f)_Q}{\|(\nabla f)_Q\|} = \langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$

3. *(15pts)* Consider the function

$$f(x,y) = x^2y + \frac{1}{9}y^3 - 2xy + 19$$

(a) Find all critical points of f. We need to solve the system

$$\begin{cases} f_x = 2y(x-1) = 0 \\ f_y = x^2 + \frac{1}{3}y^2 - 2x = 0 \end{cases}$$

But Eqn 1 implies that

$y = 0$ or $x = 1$

$y=0 \to x(x-2) = 0 \to (0,0)$, $(2,0)$

$x=1 \to \frac{1}{3}y^2 - 1 = 0 \to y = \pm\sqrt{3} \to (1,\sqrt{3})$, $(1,-\sqrt{3})$

$$\text{C.P.}(f) = \{(0,0), (2,0), (1,\sqrt{3}), (1,-\sqrt{3})\}.$$

(b) Classify the critical points of f as local maxmimum, local minimum or saddle.

Apply Second Derivative Test:

$$\Delta(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2y & 2(x-1) \\ 2(x-1) & \frac{2}{3}y \end{vmatrix} = 4\begin{vmatrix} y & x-1 \\ x-1 & \frac{1}{3}y \end{vmatrix}$$

$$= 4\left(\frac{1}{3}y^2 - (x-1)^2\right)$$

Since $\Delta(0,0) = -4 < 0$, $\Delta(2,0) = -4 < 0$, the points $(0,0)$ and $(2,0)$ are saddle points. But

$\Delta(1, \pm\sqrt{3}) = 4 > 0$, that is, $(1,\pm\sqrt{3})$ are extreme points.

$f_{xx}(1,\sqrt{3}) = 2\sqrt{3} > 0 \longrightarrow (1,\sqrt{3})$ is a local min. point

$f_{xx}(1,-\sqrt{3}) = -2\sqrt{3} < 0 \longrightarrow (1,-\sqrt{3})$ is a local max. point.

4. *(15pts)* Using the method of Lagrange Multipliers find the maximum and minimum values of the function $f(x, y, z) = x^2 + x + 2y^2 + 3z^2$ on the sphere $x^2 + y^2 + z^2 = 1$.

We have to solve the following nonlinear system

$$\begin{cases} 2x+1 = \lambda \cdot 2 \cdot x \\ 2y = \lambda \cdot y \\ 3z = \lambda \cdot z \\ x^2+y^2+z^2 = 1 \end{cases}$$

Note that $\lambda \neq 0$, for in this case $x = -1/2$, $y = 0$, $z = 0$ is a point out of the sphere.

(Eqn 2) $(\lambda - 2)y = 0 \rightarrow \lambda = 2$ or $y = 0$

$y = 0 \rightarrow$ (Eqn 3) $(\lambda - 3)z = 0$

$\lambda = 3$

(Eqn 1) $x = 1/4$, (Eqn 4) $z = \pm \frac{\sqrt{15}}{4}$;

$\downarrow$

$(1/4, 0, \pm \frac{\sqrt{15}}{4})$

$z = 0 \rightarrow$ (Eqn 4) $x = \pm 1$

(Eqn 1) $\lambda = \frac{\pm 2 + 1}{\pm 2}$.

$(\pm 1, 0, 0)$

$\lambda = 2 \longrightarrow$ (Eqn 3) $z = 0$, (Eqn 1) $x = 1/2 \longrightarrow$ (Eqn 4) $y = \pm \frac{\sqrt{3}}{2}$.

$(1/2, \pm \sqrt{3}/2, 0)$.

So, $C.P.(f) = \{ (1/4, 0, \pm \sqrt{15}/4), (\pm 1, 0, 0), (1/2, \pm \sqrt{3}/2, 0) \}$.

Moreover $f(1,0,0) = 2$, $f(-1,0,0) = 0$, $f(1/4, 0, \pm \sqrt{15}/4) = \frac{50}{16}$ and $f(1/2, \pm \sqrt{3}/2, 0) = \frac{36}{16}$. Thus f attains its abs. min. at $(-1,0,0)$ and abs. max. at the points $(1/4, 0, \pm \sqrt{15}/4)$.

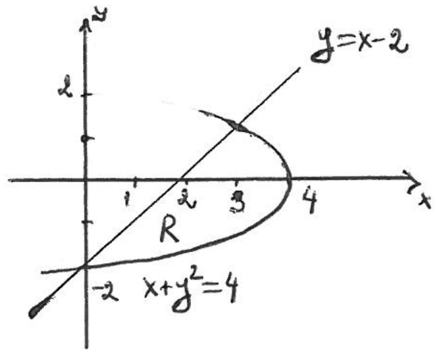
5. *(10+10+10=30pts)*

(a) Using double integrals, find the area of the region R bounded by the curves $x + y^2 = 4$ and $y = x - 2$.

$$A(R) = \iint_R dA = \int_{-2}^{1}\left(\int_{y+2}^{4-y^2} dx\right) dy = -\int_{-2}^{1}(y^2+y-2)\, dy$$

$$= 9/2 \text{, that is,}$$

$$\boxed{A(R) = 9/2}$$



(b) Reverse the order of integration in the following iterated integral. (Do not evaluate the integral)

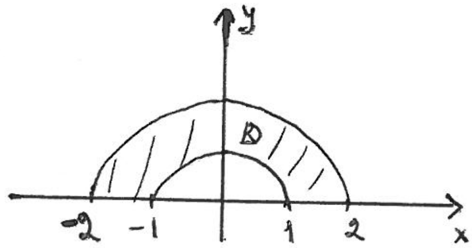
$$I = \int_{-2}^{1}\int_{y+2}^{4-y^2} \sin(x^2)\, dx\, dy$$

$$I = \int_0^3\left(\int_{-\sqrt{4-x}}^{x-2} \sin(x^2)\, dy\right) dx + \int_3^4\left(\int_{-\sqrt{4-x}}^{\sqrt{4-x}} \sin(x^2)\, dy\right) dx$$

(c) Evaluate

$$I = \iint_D \ln\sqrt{x^2 + y^2}\, dA$$

where D is the region above the x-axis between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$. (Hint: Use polar coordinates).



$$I = \int_0^{\pi}\left(\int_1^2 \ln(r)\, r\, dr\right) d\theta =$$

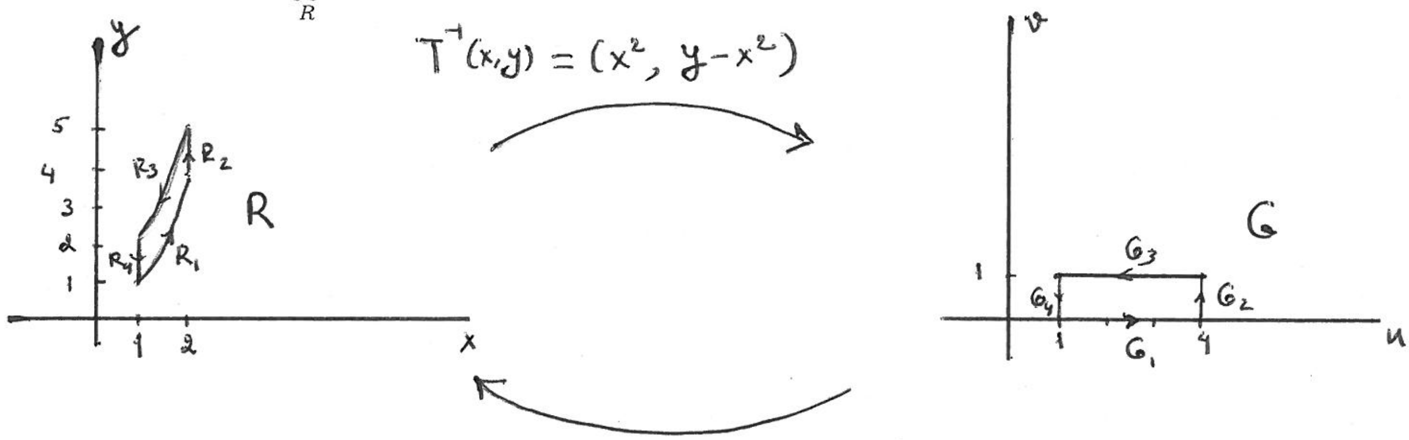
$$= \int_0^{\pi}\left(\frac{r^2}{2}\ln(r)\Big|_1^2 - \frac{1}{2}\int_1^2 r^2 \frac{1}{r}\, dr\right) d\theta$$

$$= \int_0^{\pi}\left(2\ln(2) - \frac{1}{4}3\right) d\theta = \boxed{\left(2\ln(2) - \frac{3}{4}\right)\pi}$$

6. *(10pts)*
Using the substitution

$$u = x^2, v = y - x^2$$

evaluate the integral $\iint_R 2x^3\, dA$ where R is the region bounded by $x = 1, x = 2, y = x^2, y = x^2 + 1$.



$$T(u,v) = (\sqrt{u}, u+v) \text{ for } u \geq 0.$$

Note that

$$T^{-1}(R_1) = \{(x^2, 0) : 1 \leq x \leq 2\} = \{(u,0) : 1 \leq u \leq 4\} = G_1,$$

$$T^{-1}(R_2) = \{(4, y-4) : 4 \leq y \leq 5\} = \{(4,v) : 0 \leq v \leq 1\} = G_2,$$

$$T^{-1}(R_3) = \{(x^2, 1) : 1 \leq x \leq 2\} = \{(u,1) : 1 \leq u \leq 4\} = G_3,$$

$$T^{-1}(R_4) = \{(1, y-1) : 1 \leq y \leq 2\} = \{(1,v) : 0 \leq v \leq 1\} = G_4,$$

and $J(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2\sqrt{u} & 0 \\ 1 & 1 \end{vmatrix} = \dfrac{1}{2\sqrt{u}}$, which is positive inside of G.

Using Change of Variables Formula, we derive that

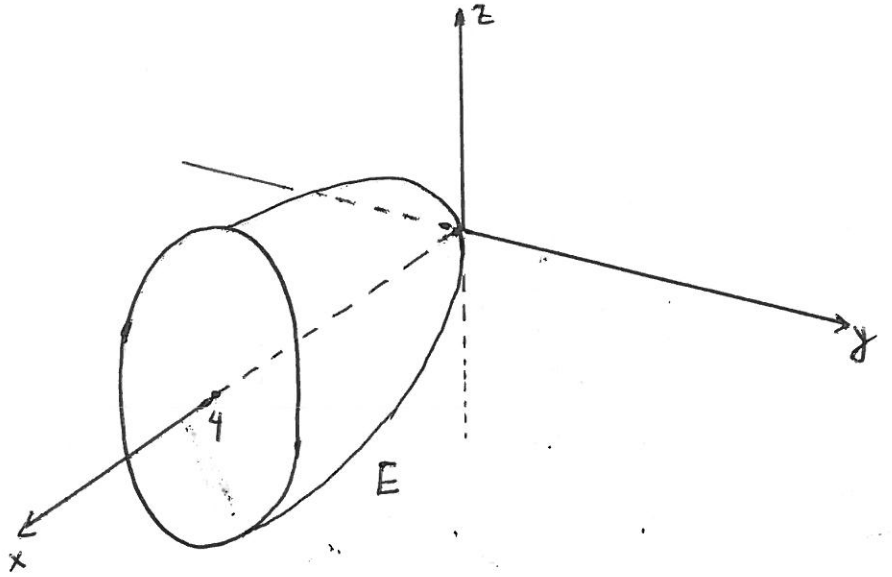
$$\iint_R 2x^3\, dA = \iint_G 2u^{3/2} \frac{1}{2\sqrt{u}}\, du\, dv = \int_0^1 \left(\int_1^4 u\, du\right) dv =$$

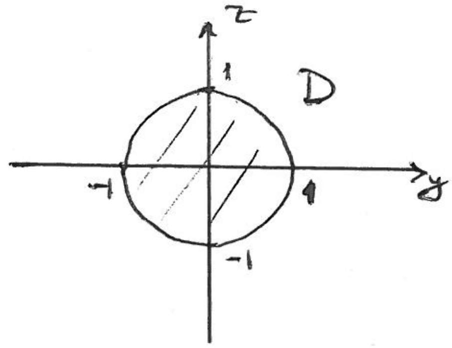
$$= \boxed{\frac{15}{2}}$$

7. *(10pts)* Find

$$\iiint_E x \, dV$$

where E is the region bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.





First note that D is the projection of the domain E onto yz-plane. Using Fubini's Theorem, we derive that

$$\iiint_E x\,dV = \iint_D \left(\int_{4y^2+4z^2}^{4} x\,dx \right) dy\,dz = \frac{1}{2} \iint_D x^2 \Big|_{4y^2+4z^2}^{4} dy\,dz$$

$$= 8 \iint_D \left(1 - (y^2+z^2)^2\right) dy\,dz = 8 \int_0^{2\pi} \int_0^1 r(1-r^4)\,dr\,d\theta$$

$$= 8 \int_0^{2\pi} \left(\tfrac{1}{2} - \tfrac{1}{6}\right) d\theta = \boxed{\tfrac{16\pi}{3}}$$